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Existence of pure Nash equilibria in discontinuous and non quasiconcave games

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Abstract

In a recent but well known paper, Reny has proved the existence of Nash equilibria for compact and quasiconcave games, with possibly discontinuous payoff functions. In this paper, we prove that the quasiconcavity assumption in Reny's theorem can be weakened: we introduce a measure allowing to localize the lack of quasiconcavity, which allows to refine the analysis of equilibrium existence.[1]

Keywords: Nash equilibrium, discontinuity, quasiconcavity.

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1 Introduction

The purpose of this paper is to relax the quasiconcavity assumption in the standard Nash equilibrium existence results. Several papers have weakened the continuity assumption of payoff functions (see, for example, Topkis (1979), Dasgupta and Maskin (1986), Baye, Tian and Zhou (1993), Reny (1999) or more recently Carmona (in Press)), with various applications, for example to Hotelling's model of price competition or to patent races. Yet, only a few papers have tried to weaken the quasiconcavity assumption, although many games in the economic literature have non quasiconcave payoff functions. Such papers could be classified in several categories, observing the method used to circumvent the non quasiconcavity:

- a first possible method is to relax directly the convexity assumption of the best reply correspondences (see, for example, Friedman and Nishimura (1981), McLennan (1989) or McClendon (2005)). Unfortunately, the properties assumed on the best reply correspondences are generally not derived from hypotheses on the payoff functions. Thus, such technique may be difficult to use in practice;

- a second method is to use the convexification of preferences when the number of players becomes sufficiently large (see, for example, Starr (1969)). A drawback of this approach is that it depends on the number of players;

- a third approach is to enlarge the definition of a pure Nash equilibrium, for instance by considering mixed-strategy equilibria, or generalized equilibria (see Kostreva (1989));

- a fourth way is to remark that the standard quasiconcavity assumption can be weakened by requiring it to hold only along the diagonal of payoffs (see, for more details, Baye, Tian and Zhou (1993) or Reny (1999));

- last, another answer to the nonconvexity issue is to look at particular classes of games, as supermodular games (see, for example, Topkis (1979)), for which the standard topological fixed point theorems can be avoided, using lattice-theoritical techniques.

In this paper, we propose another approach to obtain the existence of a (standard) Nash equilibrium in pure strategies, without assumptions on the best reply correspondences or on the number of agents, and we allow non quasiconcavity of payoff functions.

First, for every player i, we introduce a function $\rho_i : X \to \mathbb{R}$ (where X is the product of the pure strategy sets of the agents) which measures the non quasiconcavity of the payoff function of player i, and which is easy to compute for many games.

Then, using the measures ρ_i, we exhibit a condition called *strong better-reply security*, which is a reinforcement of Reny's better-reply security condition (Reny (1999)), and which provides the existence of a Nash equilibrium in pure strategies without quasiconcavity assumption. More precisely, our main condition says that for every non equilibrium strategy profile x^* and every payoff vector u^* resulting from strategies approaching x^*, some player i has a strategy yielding a payoff strictly above $u_i^* + \rho_i(x^*)$ even if the others deviate slightly from x^*. Since for quasiconcave games we obtain $\rho_i(x^*) = 0$ for every player, in this case the last condition is exactly the better-reply security assumption of Reny.

Roughly speaking, the principle of the Nash equilibrium existence proof is to associate to each game G a quasiconcavified game $\tilde{G}$ as follows: for every player i and every strategy profile x_{-i} of the other players, player i's payoff function $u_i(., x_{-i})$ is replaced by its quasiconcave hull. Then, assuming G is strongly better-reply secure, it is proved that the game $\tilde{G}$ is better-reply secure, and also that the set of equilibria of $\tilde{G}$ is equal to the set of equilibria of G, which finally yields the existence of an equilibrium from Reny's theorem.

The motivation for the weakening of the quasiconcavity assumption, aside from the fact that many games exhibit non quasiconcavity, could be also a better understanding of the existence or non existence issue of equilibria: up to now, most attention in the literature has been concentrated on the continuity problem, and one of the aim of this paper is to offer a method to refine the analysis of equilibrium existence in game theory, in particular to be able to localize the non quasiconcavity issues.

The remainder of this paper is organized as follows: in Section 2, we describe the non quasiconcavity measure and its main properties. In Section 3, we use the idea introduced in Section 2 to define our class of games, strongly better-reply secure games, which strictly contains the class of quasiconcave and better-reply secure games. Then, our main pure strategy equilibrium existence result is stated and proved. In Section 4 are described two simple conditions (payoff security and weak reciprocal upper semicontinuity), which together imply strong better-reply security. Finally, in Section 5, the previous results are extended to quasisymmetric games, for which the non quasiconcavity measure can be restricted along the diagonal of payoffs. This permits to extend some standard equilibrium existence results for quasisymmetric games, as Reny's one (1999) or Baye et al.'s one (1993), to a nonconvex framework.

2 Measure of lack of quasiconcavity

In this section, we define a measure ρ_f of lack of quasiconcavity for every real-valued function f defined on a nonempty convex subset of a topological vector space. The idea we introduce will be used in the next section to measure the lack of quasiconcavity of payoff functions. Roughly, we want to overcome the dichotomy "to be quasiconcave or not to be quasiconcave", by defining a local index of non quasiconcavity.

For every $n \in \mathbb{N}$, let Δ^{n-1} be the simplex of $\mathbb{R}^n$, defined by $\Delta^{n-1} = \{(t_1, ..., t_n) \in \mathbb{R}^n_+ : t_1 + t_2 + ... + t_n = 1\}$. Let X be a topological vector space. For every $n \in \mathbb{N}$, $t \in \Delta^{n-1}$ and $(x_1, ..., x_n) \in X^n$, we denote $t \cdot x = t_1x_1 + t_2x_2 + ... + t_nx_n$. Let Y be a convex subset of X, and consider a function $f : Y \to \mathbb{R}$. In the following, co$Y$ denotes the convex hull of Y.

We recall that f is said to be quasiconcave if for every $n \in \mathbb{N}$ and for every $(t, y) \in \Delta^{n-1} \times Y^n$, one has $f(t \cdot y) \geq \min\{f(y_1), ..., f(y_n)\}$. We would like to measure how much the previous inequality can be false at $x \in Y$. For this purpose, we introduce the mapping $\pi_f(x)$ defined as the following supremum

$$\pi_f(x) = \sup\{\min\{f(y_1), ..., f(y_n)\} - f(x)\}$$

over all $n \in \mathbb{N}$ and all families $\{y_1, ..., y_n\}$ of Y such that $x \in$co$\{y_1, ..., y_n\}$. Our final measure of lack of quasiconcavity of f is the upper semicontinuous regularization of the previous mapping:

$$\forall x \in Y,\ \rho_f(x) = \limsup_{x' \to x} \pi_f(x').$$

Definition 2.1 *The mapping ρ_f defined above is called the measure of lack of quasiconcavity of f*

Figure 1 gives an example of non quasiconcave mapping and of its measure of lack of quasiconcavity. The following lemma gathers some straightforward properties of ρ_f.

Lemma 2.1 *i) One has $\rho_f \geq \pi_f \geq 0$.*
ii) If f is bounded, then so is ρ_f.
iii) f is quasiconcave if and only if for every $x \in Y$, $\rho_f(x) = 0$.

Now, the following proposition links ρ_f to the quasiconcave envelope of f:

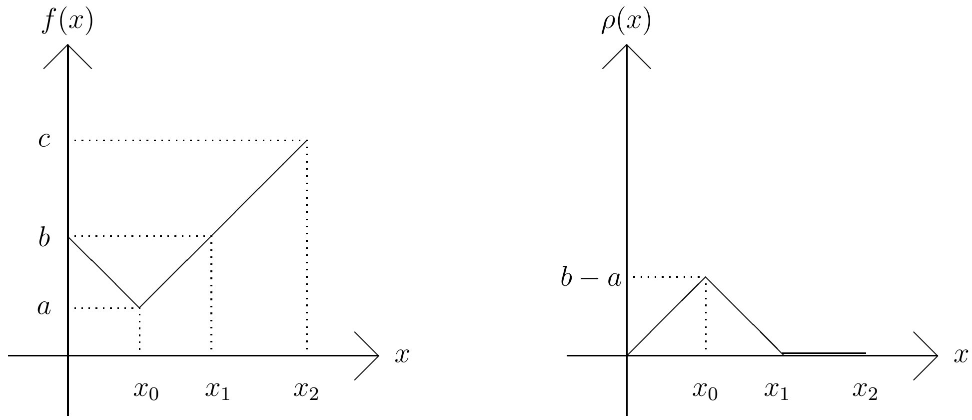


Figure 1: graph of a non quasiconcave mapping f and of its measure of lack of quasiconcavity.

Proposition 2.2 *Let $\tilde{f}$ be the quasiconcave hull of f (see [5], p.33), defined by*

$$\forall x \in Y,\ \tilde{f}(x) = \inf\{h(x) :\ h : Y \to \mathbb{R} \text{ quasiconcave},\ f \leq h\}.$$

Then, one has

$$\tilde{f}(x) = \sup\{\min\{f(y_1), ..., f(y_n)\}\}$$

over all $n \in \mathbb{N}$ and all families $\{y_1, ..., y_n\}$ of Y such that $x \in co\{y_1, ..., y_n\}$. Thus, $\pi_f(x) = \tilde{f}(x) - f(x)$.

Proof. Define $\tilde{f}_1(x)$ for every $x \in Y$, as the following supremum

$$\tilde{f}_1(x) = \sup\{\min\{f(y_1), ..., f(y_n)\}\}$$

over all $n \in \mathbb{N}$ and all families $\{y_1, ..., y_n\}$ of Y such that $x \in \text{co}\{y_1, ..., y_n\}$. Proposition 2.2 is equivalent to

$$\forall x \in Y,\ \inf\{h(x) :\ h : Y \to \mathbb{R} \text{ quasiconcave},\ f \leq h\} = \tilde{f}_1(x).$$

To prove this equality, first notice that one has clearly $f \leq \tilde{f}_1$ from the definition of $\tilde{f}_1$. Second, if $h : Y \to \mathbb{R}$ is quasiconcave with $f \leq h$, then for every $x \in Y$, every $n \in \mathbb{N}$ and every family $\{y_1, ..., y_n\}$ of Y such that $x \in \text{co}\{y_1, ..., y_n\}$, one has $\min\{f(y_1), ..., f(y_n)\} \leq \min\{h(y_1), ..., h(y_n)\} \leq h(x)$. Passing to the supremum in the above inequality, one obtains $\tilde{f}_1(x) \leq h(x)$. Third, to finish, we prove that $\tilde{f}_1$ is quasiconcave. Let $(x, y) \in Y^2$, $\lambda \in [0, 1]$ and $\varepsilon > 0$. From the definition of $\tilde{f}_1$, there exists $n \in \mathbb{N}$ and $(x_1, ..., x_n, y_1, ..., y_n) \in Y^{2n}$ such that $x \in \text{co}\{x_1, ..., x_n\}$, $y \in \text{co}\{y_1, ..., y_n\}$, and such that

$$\min\{f(x_1), ..., f(x_n)\} \geq \tilde{f}_1(x) - \varepsilon$$

and

$$\min\{f(y_1), ..., f(y_n)\} \geq \tilde{f}_1(y) - \varepsilon.$$

Now, since $\lambda x + (1-\lambda)y \in \text{co}\{x_1, ..., x_n, y_1, ..., y_n\}$, one has

$$\tilde{f}_1(\lambda x + (1-\lambda)y) \geq \min\{f(x_1), ..., f(x_n), f(y_1), ..., f(y_n)\},$$

thus one finally gets

$$\tilde{f}_1(\lambda x + (1-\lambda)y) \geq \min\{\tilde{f}_1(x), \tilde{f}_1(y)\} - \varepsilon.$$

This finishes the proof of Proposition 2.2.

Remark 2.2 *Proposition 2.2 provides another possible definition of ρ_f: it is the upper semi-continuous regularization of the distance between f and its quasiconcave hull. For a practical purpose, it is the definition we shall often use.*

3 The class of strongly better-reply secure games

The aim of this section is to define a class of non quasiconcave games for which a Nash equilibrium exists. First, in the following subsection, we extend the definition of the measure of lack of quasiconcavity to payoff functions.

3.1 Definition of a game and measure of lack of quasiconcavity of payoff functions

Consider a game with N players. The pure strategy set of each player i, denoted by X_i, is a non-empty, compact and convex subset of a topological vector space. Each agent i has a bounded payoff function

$$u_i : X = \prod_{i=1}^{N} X_i \to \mathbb{R}.$$

A game G is a couple $G = ((X_i)_{i=1}^N, (u_i)_{i=1}^N)$. Throughout this paper, a game G satisfying the above assumptions will be called a compact game.

For every $x \in X$ and every $i \in \{1, ..., N\}$, we denote $x_{-i} = (x_j)_{j \neq i}$ and $X_{-i} = \Pi_{j \neq i} X_j$. We say that the game G is quasiconcave if for every player i and every strategy $x_{-i} \in X_{-i}$, the mapping $u_i(., x_{-i})$, defined on X_i, is quasiconcave.

Recall that $x^* = (x_1^*, ..., x_N^*) \in X$ is a Nash equilibrium if for every player i, one has $u_i(x_i^*, x_{-i}^*) \geq u_i(x_i, x_{-i}^*)$ for every $x_i \in X_i$. For instance, it is well known that for every compact and quasiconcave game, if the payoff functions are continuous then there exists a Nash equilibrium.

To weaken the standard quasiconcavity assumption, we introduce the measure of lack of quasiconcavity of payoff functions as follows, using the previous section: in the following definition, for every player i and every $x = (x_i, x_{-i}) \in X$, $u_i(., x_{-i})$ denotes the mapping defined from X_i to $\mathbb{R}$ by $u_i(., x_{-i})(x_i) = u_i(x)$ for every $x_i \in X_i$.

Definition 3.1 *For every $i = 1, ..., N$ and every $x \in X$, we define the measure $\rho_i : X \to \mathbb{R}$ of lack of quasiconcavity of player i's payoff function as follows:*

$$\rho_i(x) = \limsup_{x' \to x} \pi_{u_i(., x'_{-i})}(x'_i),$$

where the definition of π is given in Section 2.

Thus, from Proposition 2.2, the measure $\rho_i : X \to \mathbb{R}$ of lack of quasiconcavity of player i's payoff function at $x = (x_i, x_{-i})$ is the upper semicontinuous regularization (with respect to the strategy profile x) at x of the distance between the quasiconcave hull $\tilde{u}_i$ of u_i (with respect to the action of player i) and u_i. Clearly, for every $x \in X$, $\rho_i(x) \geq 0$, and a compact game G is quasiconcave if and only if one has $\rho_i = 0$ for every player i. Besides, by definition, ρ_i is upper semicontinuous.

3.2 The class of better-reply secure games

Before defining our class of games, we recall the definition of better-reply secure games. This notion was introduced by Reny (1999), who has proved that every quasiconcave, compact and better-reply secure game has a Nash equilibrium, thus extending most of the previous Nash equilibrium existence results. In the following, for every profile of payoff functions $(u_i)_{i=1}^N$, $Gr(u) = \{(x, (u_1(x), ..., u_N(x)) : \ x \in X\}$ denotes the graph of the payoff functions, and $\Gamma(u) = \overline{Gr(u)}$ denotes the closure of $Gr(u)$.

Definition 3.2 *Player i can secure a payoff of $\alpha \in \mathbb{R}$ at $x = (x_i, x_{-i}) \in X$ if there exists $x'_i \in X_i$ and $V_{x_{-i}}$, an open neighborhood of x_{-i}, such that $u_i(x'_i, x'_{-i}) \geq \alpha$ for every x'_{-i} in $V_{x_{-i}}$.*

Definition 3.3 *A game G is better-reply secure if for every $(x^*, u^*) \in \Gamma(u)$ such that x^* is not a Nash equilibrium, some player i can secure a payoff strictly above u_i^*.*

3.3 The class of strongly better-reply secure games

In this subsection, we define our class of games, call strongly better-reply secure games:

Definition 3.4 *A game $G = ((X_i)_{i=1}^N, (u_i)_{i=1}^N)$ is said to be strongly better-reply secure if for every $(x^*, u^*) \in \Gamma(u)$ such that x^* is not a Nash equilibrium, some player i can secure a payoff strictly above $u_i^* + \rho_i(x^*)$.*

Remark 3.1 *Clearly, our definition strengthens Reny's Definition: every strongly better-reply secure game is better-reply secure. But the class of compact and strongly better-reply secure games strictly generalizes the class of compact, quasiconcave and better-reply secure games, as stated in the following proposition.*

Proposition 3.5 *If a game G is quasiconcave, then it is strongly better-reply secure if and only if it is better-reply secure. Moreover, there exists some compact games which are strongly better-reply secure and which are not quasiconcave.*

Proof. The first assertion is clear, since one has $\rho_i = 0$ for every quasiconcave game and every player i. To prove the second assertion, see Example 1 and Example 2 of Section 3 or Example 3 of Section 4, where are defined compact games which are strongly better-reply secure and which are not quasiconcave.

3.4 Existence of Nash equilibria in compact and strongly better-reply secure games

The purpose of this subsection is to prove our main equilibrium existence result:

Theorem 3.2 *If G is a compact and strongly better-reply secure game, then it admits a pure strategy Nash equilibrium.*

Proof. The proof[2] rests on Reny's main existence result, and is a clear consequence of the two following lemma. Hereafter, G is a strongly better-reply secure game; for every player i and every strategy $x = (x_i, x_{-i}) \in X$, $\tilde{u}_i(x_i, x_{-i})$ denotes the quasiconcave envelope of $u_i(., x_{-i})$ (with respect to x_i) at x_i, and $\tilde{G} = ((X_i)_{i=1}^N, (\tilde{u}_i)_{i=1}^N)$.

[2]The principle of the following proof was suggested by Philip Reny.

Lemma 3.3 *The set of equilibria of $\tilde{G}$ is equal to the set of equilibria of G.*

Proof. First suppose that $x \in X$ is an equilibrium of G. Then, for every player i, one has $\tilde{u}_i(x) \geq u_i(x) \geq u_i(y_i, x_{-i})$ for every strategy $y_i \in X_i$. Now, let $y_i \in X_i$, $n \in \mathbb{N}$ and $(t, z) \in \Delta^{n-1} \times (X_i)^n$ such that $y_i = \sum_{k=1}^{n} t_k z_k$. From the previous inequality, one has

$$\tilde{u}_i(x) \geq \min\{u_i(z_1, x_{-i}), ..., u_i(z_n, x_{-i})\},$$

which entails

$$\tilde{u}_i(x) \geq \sup_{n\in\mathbb{N},\ (t,z)\in\Delta^{n-1}\times X_i^n,\ t\cdot z=y_i} \min\{u_i(z_1, x_{-i}), ..., u_i(z_n, x_{-i})\} = \tilde{u}_i(y_i, x_{-i}),$$

which proves that x is an equilibrium of $\tilde{G}$.

Now suppose that $x \in X$ is an equilibrium of $\tilde{G}$, and suppose that it is not an equilibrium of G. From the strong better-reply security assumption, and since $(x, u(x)) \in \Gamma(u)$, there exists a player i, a neighborhood $V_{x_{-i}}$ of x_{-i} and a strategy $\overline{x}_i \in X_i$ of player i such that one has

$$\forall x'_{-i} \in V_{x_{-i}},\ u_i(\overline{x}_i, x'_{-i}) > u_i(x) + \rho_i(x).$$

Recalling that $\rho_i(x)$ is the u.s.c. regularization of $\tilde{u}_i(x) - u_i(x)$ at x, one has $\rho_i(x) \geq \tilde{u}_i(x) - u_i(x)$. From this inequality and from the above inequality for $x'_{-i} = x_{-i}$, one obtains

$$\tilde{u}_i(\overline{x}_i, x_{-i}) \geq u_i(\overline{x}_i, x_{-i}) > u_i(x) + (\tilde{u}_i(x) - u_i(x)) = \tilde{u}_i(x)$$

which is a contradiction with the fact that $x \in X$ is an equilibrium of $\tilde{G}$. Thus, x is an equilibrium of G

Lemma 3.4 *If G is strongly better-reply secure, then the game $\tilde{G}$ is better-reply secure.*

Proof. To prove Lemma 3.4, suppose that G is strongly better-reply secure; let $(\overline{x}, \overline{\tilde{u}}) \in \Gamma(\tilde{u})$ such that $\overline{x}$ is not an equilibrium of $\tilde{G}$. Let $\Pi_1 : X \times \mathbb{R}^N \to X$ be defined by $\Pi_1(x, u) = x$ for every $(x, u) \in X \times \mathbb{R}^N$. Let

$$\overline{u} \in \bigcap_{(\overline{x},\overline{\tilde{u}})\in V} \overline{u(\Pi_1(V \cap Gr(\tilde{u})))},$$

where the intersection is taken over all open neighborhoods V, in $\Gamma(\tilde{u})$, of $(\overline{x}, \overline{\tilde{u}})$. Note that this intersection is nonempty, because u is bounded and $(\overline{u(\Pi_1(V \cap Gr(\tilde{u})))})_{(\overline{x},\overline{\tilde{u}})\in V}$ is a family of compact subsets of $\mathbb{R}^N$ that has the finite intersection property. In a metric space context, $\overline{u}$ can be seen as the limit of a sequence $(u(x_n))_{n\in\mathbb{N}}$, where $(x_n, \tilde{u}(x_n))_{n\in\mathbb{N}}$ is a sequence converging to $(\overline{x}, \overline{\tilde{u}})$.

Since G is strongly better-reply secure, one has $u_i(x'_i, x'_{-i}) > \overline{u}_i + \rho_i(\overline{x}) + \varepsilon$ for some $\varepsilon > 0$, some player i, some $x'_i \in X_i$ and for all x'_{-i} in some neighborhood $V_{\overline{x}_{-i}}$ of $\overline{x}_{-i}$. Fix now $x'_{-i} \in V_{\overline{x}_{-i}}$.

From the definition of $\overline{u}$, and since $\rho_i(\overline{x})$ is the u.s.c. regularization of $\tilde{u}_i(x) - u_i(x)$ at $\overline{x}$, there exists $(y, \tilde{u}(y)) \in V$, where V is some open neighborhood in $\Gamma(\tilde{u})$ of $(\overline{x}, \overline{\tilde{u}})$, such that one has

$$u_i(x'_i, x'_{-i}) > \overline{u}_i + (\tilde{u}_i(y) - u_i(y)) + \frac{\varepsilon}{2}, \tag{1}$$

$$| u_i(y) - \overline{u}_i |< \frac{\varepsilon}{8}, \tag{2}$$

and

$$| \tilde{u}_i(y) - \overline{\tilde{u}}_i) |< \frac{\varepsilon}{8}. \tag{3}$$

Thus, from the above equations, one obtains $u_i(x'_i, x'_{-i}) > u_i(y) + (\overline{\tilde{u}}_i - u_i(y)) + \frac{\varepsilon}{4} = \overline{\tilde{u}}_i + \frac{\varepsilon}{4}$, which also implies $\tilde{u}_i(x'_i, x'_{-i}) > \overline{\tilde{u}}_i + \frac{\varepsilon}{4}$. This prove that $\tilde{G}$ is better-reply secure, and finishes the proof of Lemma 3.4.

If G is quasiconcave, Theorem 3.2 is exactly Reny's Theorem. Thus, Theorem 3.2 gives a simple characterization (thanks to strong better-reply security assumption) that guarantees that the quasiconcavified game $\tilde{G}$ is better-reply secure, and has the same equilibria as the game G. Besides, this characterization measures precisely the "cost"(through the measure of non quasiconcavity ρ) that has to be paid to restore equilibrium existence. Two examples illustrate Theorem 3.2:

Example 1. Consider the following game G: there are two players $i = 1, 2$; the strategy sets of each player are $X_1 = [0, V_1]$ and $X_2 = [0, V_2]$, where $V_1 > 0$ and $V_2 > 0$; the payoff functions are defined as follows, where $-i$ denotes 2 if $i = 1$ and 1 if $i = 2$: for every $(x_i, x_{-i}) \in X_i \times X_{-i}$,

$$u_i(x_i, x_{-i}) = \begin{cases} -x_i, & \text{if } x_i < x_{-i} \\ V_i - x_i, & \text{if } x_i \geq x_{-i} \end{cases}$$

Clearly, G is not quasiconcave (see Figure 2) but is compact. To compute the measure of lack of quasiconcavity ρ_i, from Proposition 2.2, we only have to find $\tilde{u}_i(., x_{-i})$, the envelop of $u_i(., x_{-i})$ ($x_{-i} \in X_{-i}$ being fixed). Then, $\rho_i(x_i, x_{-i})$ is the upper semicontinuous regularization (with respect to $x = (x_i, x_{-i})$) of $\tilde{u}_i(x) - u_i(., x_{-i})$. See Figure 2 and Figure 3 for a representation of $u_i(., x_{-i})$, $\tilde{u}_i(., x_{-i})$ and $\rho_i(., x_{-i})$.

Now, to prove that G is strongly better-reply secure, let $(x_1^*, x_2^*, u_1^*, u_2^*) \in \Gamma(u)$ such that (x_1^*, x_2^*) is not an equilibrium. Thus, $x_1^* \neq x_2^*$, because for every $a \in [0, \min\{V_1, V_2\}]$, (a, a) is a Nash equilibrium of G. Without any loss of generality, one can suppose that $x_1^* < x_2^*$. Consequently, $\rho_2(x^*) = 0$ and $x_1^* < V_2$. Let $\varepsilon > 0$ such that $x_2^* - \varepsilon > x_1^*$. By playing $x_2 = x_2^* - \varepsilon$, player 2 obtains $V_2 - x_2^* + \varepsilon$. Since $u_2^* + \rho_2(x^*) = V_2 - x_2^*$, it proves that player 2 can secure a payoff strictly above $u_2^* + \rho_2(x^*)$ by playing x_2 (because the payoff of player 2 moves continuously when the strategy $x_1^* \neq x_2$ of player 1 is slightly modified).

In the next example, we provide a continuous and compact game which is not quasiconcave, but which is strongly better-reply secure:

Example 2. Consider the following location game G: there are two players $i = 1, 2$; the strategy sets of each player are $X = Y = [0, 1]$; for every strategy x of player 1 and every strategy y of player 2, the payoff functions are defined as follows:

$$u_1(x, y) = - | x - y |$$

$$u_2(x, y) = (\frac{1}{2} - x) | x - y |$$

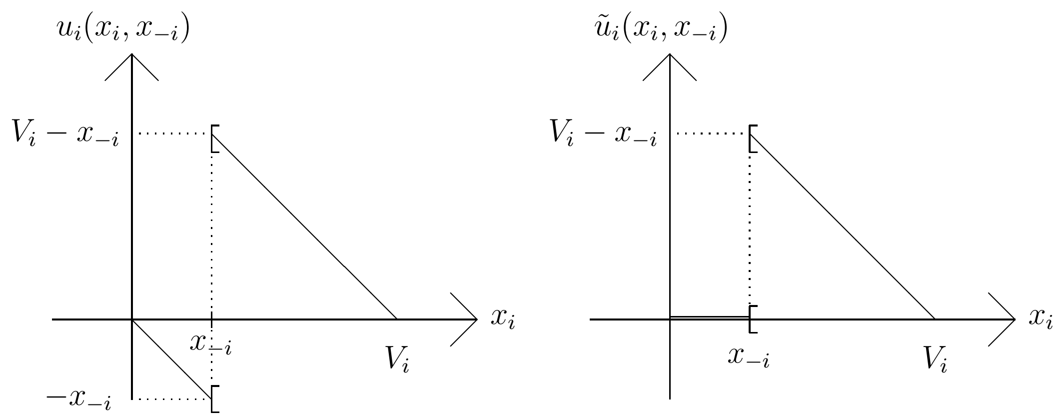


Figure 2: Graph of $u_i(., x_{-i})$ and $\tilde{u}_i(., x_{-i})$ in Example 1.

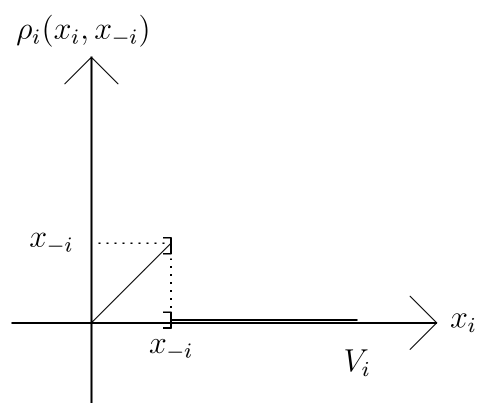


Figure 3: Graph of $\rho_i(., x_{-i})$ in Example 1.

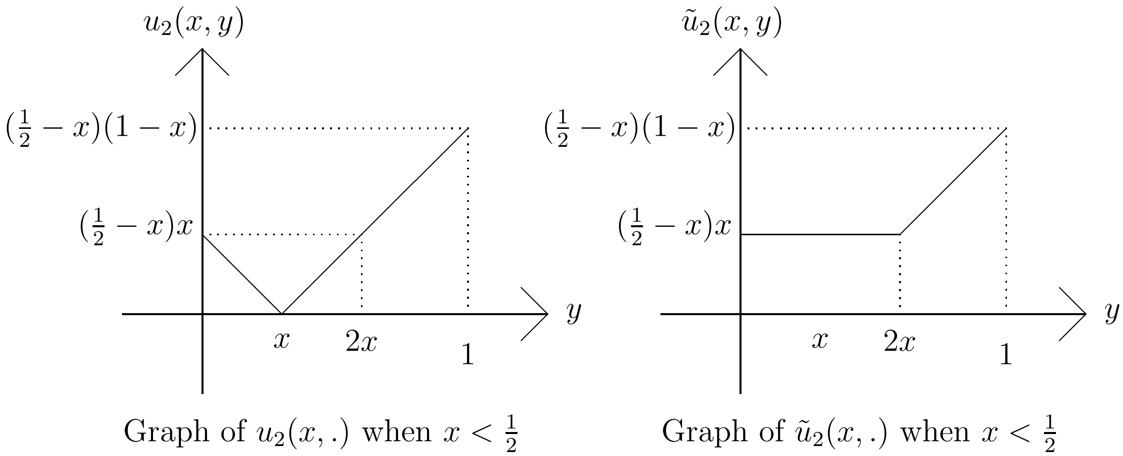


Figure 4: Graph of $u_2(x,.)$ and of $\tilde{u}_2(x,.)$ when $x<\frac{1}{2}$.

In this game, player 1 would like to choose the same location as player 2, whereas the behaviour of player 2 depends on the location of player 1: he would like to be far from player 1 if $x<\frac{1}{2}$, would like to be close to player 1 if $x>\frac{1}{2}$, and does not care for $x=\frac{1}{2}$.

The game G is not quasiconcave, because $u_2(x,.)$ is not quasiconcave for $x<\frac{1}{2}$ (see figure 4). More precisely, since $u_1(.,y)$ is quasiconcave for every $y\in Y$, one has $\rho_1=0$, and we now compute ρ_2 to measure the lack of quasiconcavity of this game. Figure 4 represents the graph of $u_2(x,.)$ and of its quasiconcave envelop $\tilde{u}_2(x,.)$ for $x<\frac{1}{2}$ fixed; from Proposition 2.2, and since the payoff functions are continuous, one has $\rho_2(x,y)=\tilde{u}_2(x,y)-u_2(x,y)$. Moreover, for $x\geq\frac{1}{2}$, $u_2(x,.)$ is quasiconcave, thus $\rho_2(x,.)=0$ in this case.

Now, to prove that G is strongly better-reply secure, let $(x^*,y^*,u_1^*,u_2^*)\in\Gamma(u)$ such that (x^*,y^*) is not an equilibrium. First notice that if $x^*\neq y^*$, then player 1, whose payoff function is continuous and quasiconcave with respect to x, can strictly secure a payoff of $u_1^*+\rho_1(x^*)=u_1(x^*)=-\mid x^*-y^*\mid$, by playing y^*. Thus, now suppose that $x^*=y^*$. Since (a,a) is an equilibrium for every $a\in[\frac{1}{2},1]$, one has $x^*<\frac{1}{2}$. This implies $\rho_2(x^*,x^*)=(\frac{1}{2}-x^*)x^*$. Consequently, player 2 can strictly secure $u_2^*+\rho_2(x^*,x^*)=(\frac{1}{2}-x^*)x^*$ by playing $2x^*+\varepsilon\in]0,1[$ for $\varepsilon>0$ small enough: indeed, it gives him a payoff of $(\frac{1}{2}-x^*)(x^*+\varepsilon)$. Thus, G is strongly better-reply secure.

4 The class of weakly reciprocal upper semicontinuous games

Reny (1999) has introduced two simple conditions, called payoff security and reciprocal upper semicontinuity (rusc), which together imply better-reply security. An advantage of these conditions is that they can be checked without any reference to the set of Nash equilibria, contrarily to better-reply security. Following the idea of the last section, we propose[3] a natural generalization of reciprocal upper semicontinuity in a non quasiconcave framework.

First recall the definitions of weak reciprocal upper semicontinuity.[4] In the following, if A and B are two sets, we let $A\setminus B=\{x\in A:\ x\notin B\}$.

Definition 4.1 *A game $G=((X_i)_{i=1}^N,(u_i)_{i=1}^N)$ is weakly reciprocal upper semicontinuous*

[3] I am grateful to two anonymous referees for suggesting me to adapt the results of the previous section for the class of payoff secure and reciprocal upper semicontinuous games.

[4] This property, very similar to rusc, and which is implied by rusc, was introduced by Bahg and Jofre (2006), who have noticed that every wrusc and payoff secure game was better-reply secure.

(wrusc) if for any $(x^*, u^*) \in \Gamma(u) \setminus Gr(u)$, there is a player i and $\hat{x}_i \in X_i$ such that $u_i(\hat{x}_i, x^*_{-i}) > u^*_i$.

Definition 4.2 *A game $G = ((X_i)_{i=1}^N, (u_i)_{i=1}^N)$ is payoff secure if for every player i, for every $x \in X$ and for every $\varepsilon > 0$, player i can secure a payoff of $u_i(x) - \varepsilon$, which means that there exists $\hat{x}_i \in X_i$ such that $u_i(\hat{x}_i, x'_{-i}) \geq u_i(x) - \varepsilon$ for all x'_{-i} in some neighborhood of x_{-i}.*

We now generalize wrusc to (possibly) non quasiconcave games:

Definition 4.3 *A (possibly non quasiconcave) game $G = ((X_i)_{i=1}^N, (u_i)_{i=1}^N)$ is weakly reciprocal upper semicontinuous (wrusc) if for any $(x^*, u^*) \in \Gamma(u) \setminus Gr(u - \rho)$, there is a player i and $\hat{x}_i \in X_i$ such that $u_i(\hat{x}_i, x^*_{-i}) > u^*_i + \rho_i(x^*)$.*

For example, let G be a game whose payoff functions are continuous and satisfying the following property: for every $x^* \in X$ with $\rho(x^*) \neq 0$, there is a player i and $\hat{x}_i \in X_i$ such that $u_i(\hat{x}_i, x^*_{-i}) > u_i(x^*) + \rho_i(x^*)$. Then, it is payoff secure and wrusc.

Proposition 4.4 *If G is payoff secure and weakly reciprocal upper semicontinuous, then it is strongly better-reply secure.*

Proof. The proof is a simple adaptation of the proof of Proposition 1 in [1]: take $(x^*, u^*) \in \Gamma(u)$, where x^* is not an equilibrium. If $(x^*, u^*) \in \Gamma(u) \setminus Gr(u - \rho)$, then wrusc implies that there is a player i and $\hat{x}_i \in X_i$ such that $u_i(\hat{x}_i, x^*_{-i}) > u^*_i + \rho_i(x^*)$; clearly, payoff security at $(\hat{x}_i, x^*_{-i})$ implies that player i can secure a payoff strictly above $u^*_i + \rho_i(x^*)$. Now, suppose that $(x^*, u^*) \in Gr(u - \rho)$. Then, for all i, $u^*_i = u_i(x^*) - \rho_i(x^*)$. Since x^* is not an equilibrium, there is a player i and $\hat{x}_i \in X_i$ such that $u_i(\hat{x}_i, x^*_{-i}) > u_i(x^*) = u^*_i + \rho_i(x^*)$, and again payoff security implies that player i can secure a payoff strictly above $u^*_i + \rho_i(x^*)$.

Note that there exist some strongly better-reply secure games which are not weakly reciprocal upper semicontinuous: indeed, in Example 1, for $a \in]0, \min\{V_1, V_2\}[$, one has $((a, a), (V_1 - a, V_2 - a)) \in \Gamma(u) \setminus Gr(u - \rho)$, but for every player $i = 1, 2$, there is no $\hat{x}_i \in X_i$ such that $u_i(\hat{x}_i, a) > V_i - a + \rho_i(a, a) = V_i$.

Example 3. Consider the following game G: there are two players $i = 1, 2$; the strategy sets of each player are $X = Y = [-1, 1]$; given the strategy $x \in X$ of player 1 and the stratey $y \in Y$ of player 2, the payoffs are defined as follows:

$$u_1(x, y) = -x$$

$$u_2(x, y) = xy^2$$

This game is continuous, so it is payoff secure, but not quasiconcave. We now prove that it is wrusc. Let $(x^*, y^*) \in X \times Y$ such that $\rho(x^*) \neq 0$. Clearly, it implies $x^* > 0$, and one has $u_1(-1, y^*) = 1 > u_1(x^*, y^*) + \rho_1(x^*, y^*) = -x^*$. It finally proves that G is wrusc (see the remark following Definition 4.3).

5 Symmetric equilibria

In this section, we improve the conclusion of the previous section, by considering the more restricted class of quasisymmetric games. Recall that a game $G = ((X_i)_{i=1}^N, (u_i)_{i=1}^N)$ is quasisymmetric if $X_1 = X_2 = ... = X_N$ and if $u_1(x, y, y, ..., y) = u_2(y, x, y, y, ..., y) = ... = u_N(y, y, ..., y, x)$ for every $x \in X_1$ and for every $y \in X_1$. For $N = 2$, a quasisymmetric game is called a symmetric game. In the following, we let $X = X_1$, and the quasisymmetric game will be denoted $G = (X, (u_i)_{i=1}^N)$. In such games, one can define the diagonal payoff function $v : X \to \mathbb{R}$ by $v(x) = u_1(x, ..., x)$ for every $x \in X$.

First of all, we define a measure of non quasiconcavity for quasisymmetric games:

Definition 5.1 *Let $G = (X, (u_i)_{i=1}^N)$ be a quasisymmetric game. For every $x \in X$, we define the measure $\rho : X \to \mathbb{R}$ of lack of quasiconcavity of G at x as follows:*

$$\rho(x) = \limsup_{x' \to x}(-v(x') + \sup_{n \in \mathbb{N},\ (t,y) \in \Delta^{n-1} \times X^n,\ t \cdot y = x'} \min\{u_1(y_1, x', ..., x'), ..., u_1(y_n, x', ..., x')\})$$

It is worthwhile to note that in the above definition, one can replace player 1 by any player withouth changing the value of ρ. Moreover, from Proposition 2.2, one can relate the previous measure to the notion of quasiconcave envelop as follows: for every $x' \in X$, define the quasiconcave envelop of $u_1(., x', ..., x')$ with respect to the first variable, denoted $\tilde{u}_1(., x', x' ..., x')$. Then, one has

$$\rho(x) = \limsup_{x' \to x}(\tilde{u}_1(x', x', x' ..., x') - u_1(x', ..., x')) \tag{4}$$

Now, recall that G is said to be diagonally quasiconcave (see Reny (1999)) if X is convex and if for each $x' \in X$ and $y_1, ..., y_n$ in X such that $x' \in \mathrm{co}\{y_1, ..., y_n\}$, one has

$$-v(x') + \min\{u_1(y_1, x', ..., x'), ..., u_1(y_n, x', ..., x')\} \leq 0.$$

Thus, if X is convex, then G is diagonally quasiconcave if and only if $\rho(x) = 0$ for every $x \in X$.

Following Reny (1999), we say that player i secures a payoff of $\alpha \in \mathbb{R}$ along the diagonal at $(x, x, ..., x) \in X^N$ if there exists $\overline{x} \in X$ such that $u_i(x', x', ..., \overline{x}, x', ..., x') \geq \alpha$ for all x' in some neighborhood of $x \in X$. Note that if G is quasisymmetric, then player i secures a payoff of $\alpha \in \mathbb{R}$ along the diagonal at $(x, x, ..., x) \in X^N$ if and only if player j secures a payoff of $\alpha \in \mathbb{R}$ along the diagonal at $(x, x, ..., x) \in X^N$, for every $j = 1, ..., N$.

We now adapt Definition 3.4 to quasisymmetric games:

Definition 5.2 *A quasisymmetric game $G = (X, (u_i)_{i=1}^N)$ is strongly diagonally better-reply secure if whenever $(x^*, v^*) \in X \times \mathbb{R}$ is in the closure of the graph of its diagonal payoff function and $(x^*, ..., x^*)$ is not an equilibrium, some player i can secure a payoff strictly above $v^* + \rho(x^*)$ along the diagonal at $(x^*, ..., x^*)$.*

If G is diagonally quasiconcave, we have seen that $\rho = 0$: in this case, the previous definition is exactly the definition of diagonally better-reply secure games, introduced by Reny. Note also that since G is quasisymmetric, in the definition above, "some player i" can be replaced by "every player i" without altering this definition.

The following theorem is an extension of Theorem 3.2 to quasisymmetric games, and can be proved similarly by considering the game $\tilde{G}$ associated to G.

Theorem 5.1 *If $G = (X, (u_i)_{i=1}^N)$ is quasisymmetric, compact and strongly diagonally better-reply secure, then it admits a symmetric pure Nash equilibrium.*

We now give an example of quasisymmetric, compact, strongly diagonally better-reply secure game which is not diagonally quasiconcave:

Example 4 In their paper, Baye et al. (1993) introduce the following game G: two duopolists with zero costs set prices $(p_1, p_2) \in [0, T] \times [0, T]$, where $T > 0$. For $i = 1, 2$, the payoff functions are

$$\overline{u}_i(p_i, p_{-i}) = \begin{cases} p_i, & \text{if } p_i \leq p_{-i} \\ p_i - c, & \text{otherwise} \end{cases}$$

where $0 < c < T$, and where $-i$ denotes 1 if $i = 2$ and 2 if $i = 1$. The interpretation is that each firm pays brand-loyal consumers a penalty of c if the other firm beats its price. It is easy to prove that this game is symmetric, compact, diagonally quasiconcave and diagonally better-reply secure. Thus, one could apply Reny (1999) or the main result of Baye et al., in order to obtain the existence of a pure Nash equilibrium.

Now, let $\varepsilon \in]0, \min\{c, T - c\}[$, and consider the following modification G_ε of the previous game: suppose that the penalty of firm i is reinforced for $p_{-i} \leq T - c$: in this case, firm i pays brand-loyal consumers a penalty of c if $p_{-i} < p_i + \varepsilon$, and nothing otherwise. Thus, firm i may have to pay the penalty even if p_{-i} is (slightly) larger than p_i. Consequently, one can write the modified payoff function as follows:

$$u_i(p_i, p_{-i}) = \begin{cases} p_i, & \text{if } p_i \leq p_{-i} - \varepsilon(\mathbf{1}_{p_{-i} \leq T-c}(p_{-i})) \\ p_i - c, & \text{otherwise} \end{cases}$$

where $\mathbf{1}_{p_{-i} \leq T-c}(p_{-i}) = 1$ if $p_{-i} \leq T - c$ and $\mathbf{1}_{p_{-i} \leq T-c}(p_{-i}) = 0$ otherwise. Note that for $\varepsilon = 0$, one has $G_\varepsilon = G$. Besides, clearly, G_ε is symmetric and compact. We now prove that it is strongly diagonally better-reply secure. First, we compute ρ, the measure of non quasiconcavity of G_ε. For this purpose, recall that for every $p_2 \in [0, T]$, $\tilde{u}_1(., p_2)$ denotes the quasiconcave envelop of the mapping $u_1(., p_2)$ with respect to the first variable, defined in Section 2. Now, consider the three following cases:

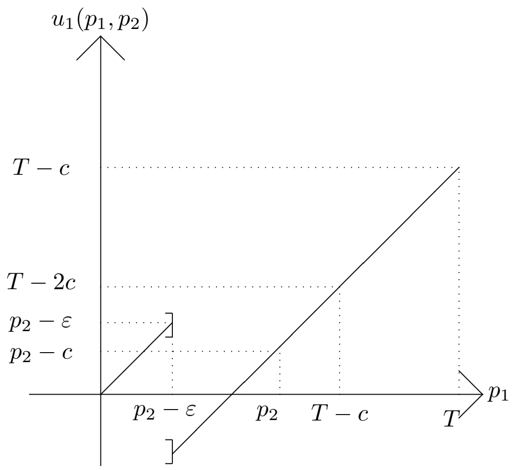
i) Suppose $p_2 < \varepsilon$, which implies $p_2 < T - c$. Then, one has $u_1(p_1, p_2) = p_1 - c$ for every $p_1 \in [0, T]$. Thus, $u_1(., p_2)$ is quasiconcave, and $\tilde{u}_1(p_1, p_2) = p_1 - c$ for every $p_1 \in [0, T]$.

ii) Suppose $p_2 \in [\varepsilon, T - c]$. One has $u_1(p_1, p_2) = p_1$ if $p_1 \leq p_2 - \varepsilon$ and $u_1(p_1, p_2) = p_1 - c$ if $p_1 > p_2 - \varepsilon$. Thus (see figure 5), one has $\tilde{u}_1(p_1, p_2) = p_1$ if $p_1 \leq p_2 - \varepsilon$, $\tilde{u}_1(p_1, p_2) = p_2 - \varepsilon$ if $p_1 \in [p_2 - \varepsilon, p_2 + c - \varepsilon]$ and $\tilde{u}_1(p_1, p_2) = p_1 - c$ if $p_1 > p_2 + c - \varepsilon$.

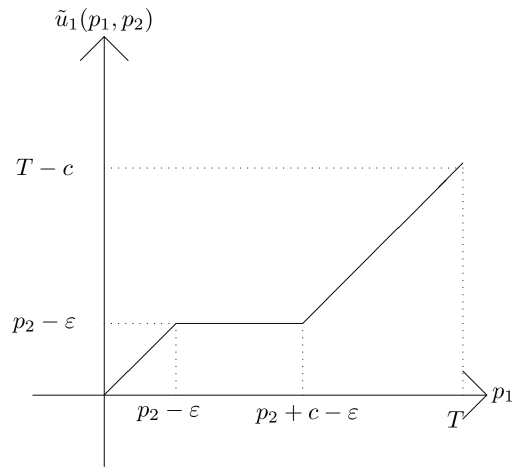
iii) Last, suppose $p_2 > T - c$. One has $u_1(p_1, p_2) = p_1$ if $p_1 \leq p_2$, and $u_1(p_1, p_2) = p_2 - c$ if $p_1 > p_2$. Thus, one has $\tilde{u}_1(p_1, p_2) = p$ if $p_1 \leq p_2$, and $\tilde{u}_1(p_1, p_2) = T - c$ if $p_1 > p_2$.

From the three previous cases, one has $\tilde{u}_1(p, p) - u_1(p, p) = 0$ for every $p < \varepsilon$, $\tilde{u}_1(p, p) - u_1(p, p) = c - \epsilon$ for every $p \in [\varepsilon, T - c]$ and $\tilde{u}_1(p, p) - u_1(p, p) = 0$ for every $p > T - c$. Thus, from Equation 4, one obtains $\rho(p) = 0$ for every $p < \varepsilon$, $\rho(p) = c - \varepsilon$ for every $p \in [\varepsilon, T - c]$ and $\rho(p) = 0$ for every $p > T - c$.

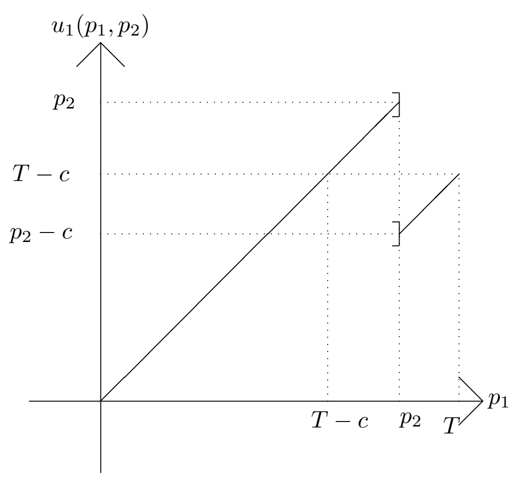
Eventually, to prove that G_ε is strongly diagonally better-reply secure, consider (p^*, v^*) in the closure of the graph of its diagonal payoff function, such that (p^*, p^*) is not an equilibrium. Thus $p^* \leq T - c$, because for every $p > T - c$, (p, p) is an equilibrium (see Figure 5). Now, if $p^* < \varepsilon$ then consumer 1 can strictly secure $v^* + \rho(p^*) = v^* = p^* - c$ by playing strictly above p^*. On the other hand, if $p^* \in [\varepsilon, T - c]$, then consumer 1 can strictly secure $v^* + \rho(p^*) = p^* - c + c - \varepsilon = p^* - \varepsilon$ by playing $p^* + c$. So, G_ε is strongly diagonally better-reply secure.



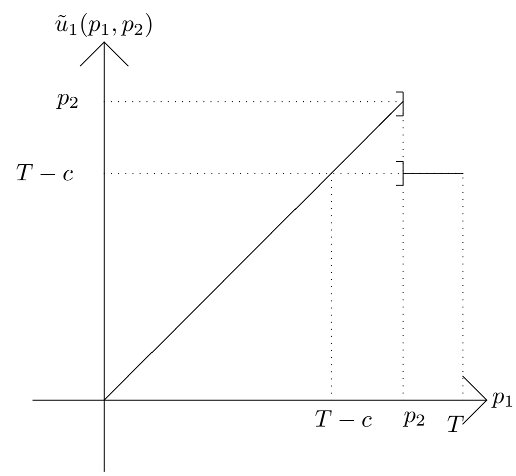
Graph of $u_1(.,p_2)$ when $\varepsilon \leq p_2 \leq T-c$.



Graph of the quasi-concave envelop of $u_1(.,p_2)$ when $\varepsilon \leq p_2 \leq T-c$



Graph of $u_1(.,p_2)$ when $p_2 > T-c$.



Graph of of the quasi-concave envelop of $u_1(.,p_2)$ when $p_2 > T-c$.

Figure 5: graph of $u_1(.,p_2)$ and $\tilde{u}_1(.,p_2)$ in Example 4

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